

# Locally finite groups in which every non-cyclic subgroup is self-centralizing

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## Abstract

Locally finite groups having the property that every non-cyclic subgroup contains its centralizer are completely classified.

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## 1. Introduction

A subgroup  $H$  of a group  $G$  is *self-centralizing* if the centralizer  $C_G(H)$  is contained in  $H$ . In [1] it has been remarked that a locally graded group in which all non-trivial subgroups are self-centralizing has to be finite; therefore it has to be either cyclic of prime order or non-abelian of order being the product of two different primes.

In this article, we consider the more extensive class  $\mathfrak{X}$  of all groups in which every non-cyclic subgroup is self-centralizing. In what follows we use the term  $\mathfrak{X}$ -groups in order to denote groups in the class  $\mathfrak{X}$ . The study of properties of  $\mathfrak{X}$ -groups was initiated in [1]. In particular, the first four authors determined the structure of finite  $\mathfrak{X}$ -groups which are either nilpotent, supersoluble or simple.

In this paper, Theorem 2.1 gives a complete classification of finite  $\mathfrak{X}$ -groups. We remark that this result does not depend on classification of the finite simple groups rather only on the classification of groups with dihedral or semidihedral Sylow 2-subgroups. We also determine the infinite soluble  $\mathfrak{X}$ -groups, and the infinite locally finite  $\mathfrak{X}$ -groups, the results being presented in Theorems 3.6 and 3.7. It turns out that these latter groups are suitable finite extensions either of the infinite cyclic group or of a Prüfer  $p$ -group,  $\mathbb{Z}_{p^\infty}$ , for some prime

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$p$ . Theorem 3.7 together with Theorem 2.1 provides a complete classification of locally finite  $\mathfrak{X}$ -groups.

We follow [2] for basic group theoretical notation. In particular, we note that  $F^*(G)$  denotes the generalized Fitting subgroup of  $G$ , that is the subgroup of  $G$  generated by all subnormal nilpotent or quasisimple subgroups of  $G$ . The latter subgroups are the components of  $G$ . We see from [2, Section 31] that distinct components commute. The fundamental property of the generalized Fitting subgroup that we shall use is that it contains its centralizer in  $G$  [2, (31.13)]. We denote the alternating group and symmetric group of degree  $n$  by  $\text{Alt}(n)$  and  $\text{Sym}(n)$  respectively. We use standard notation for the classical groups. The notation  $\text{Dih}(n)$  denotes the dihedral group of order  $n$  and  $\text{Q}_8$  is the quaternion group of order 8. The term quaternion group will cover groups which are often called generalized quaternion groups. The cyclic group of order  $n$  is represented simply by  $n$ , so for example  $\text{Dih}(12) \cong 2 \times \text{Dih}(6) \cong 2 \times \text{Sym}(3)$ . Finally  $\text{Mat}(10)$  denotes the Mathieu group of degree 10. The Atlas [3] conventions are used for group extensions. Thus, for example,  $p^2:\text{SL}_2(p)$  denotes the split extension of an elementary abelian group of order  $p^2$  by  $\text{SL}_2(p)$ .

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## 2. Finite $\mathfrak{X}$ -groups

In this section we determine all the finite groups belonging to the class  $\mathfrak{X}$ . The main result is the following.

**Theorem 2.1.** *Let  $G$  be a finite  $\mathfrak{X}$ -group. Then one of the following holds:*

- (1) *If  $G$  is nilpotent, then either*
  - (1.1)  *$G$  is cyclic;*
  - (1.2)  *$G$  is elementary abelian of order  $p^2$  for some prime  $p$ ;*
  - (1.3)  *$G$  is an extraspecial  $p$ -group of order  $p^3$  for some odd prime  $p$ ; or*
  - (1.4)  *$G$  is a dihedral, semidihedral or quaternion 2-group.*
- (2) *If  $G$  is supersoluble but not nilpotent, then, letting  $p$  denote the largest prime divisor of  $|G|$  and  $P \in \text{Syl}_p(G)$ , we have that  $P$  is a normal subgroup of  $G$  and one of the following holds:*
  - (2.1)  *$P$  is cyclic and either*
    - (2.1.1)  *$G \cong D \rtimes C$ , where  $C$  is cyclic,  $D$  is cyclic and every non-trivial element of  $D$  acts fixed point freely on  $C$  (so  $G$  is a Frobenius group);*
    - (2.1.2)  *$G = D \rtimes C$ , where  $C$  is a cyclic group of odd order,  $D$  is a quaternion group, and  $C_G(C) = C \times D_0$  where  $D_0$  is a cyclic subgroup of index 2 in  $D$  with  $G/D_0$  a dihedral group; or*

- (2.1.3)  $G = D \ltimes C$ , where  $D$  is a cyclic  $q$ -group,  $C$  is a cyclic  $q'$ -group (here  $q$  denotes the smallest prime dividing the order of  $G$ ),  $1 < Z(G) < D$  and  $G/Z(G)$  is a Frobenius group;
- (2.2)  $P$  is extraspecial and  $G$  is a Frobenius group with cyclic Frobenius complement of odd order dividing  $p - 1$ .
- (3) If  $G$  is not supersoluble and  $F^*(G)$  is nilpotent, then either (3.1) or (3.2) below holds.
  - (3.1)  $F^*(G)$  is elementary abelian of order  $p^2$ ,  $F^*(G)$  is a minimal normal subgroup of  $G$  and one of the following holds:
    - (3.1.1)  $p = 2$  and  $G \cong \text{Sym}(4)$  or  $G \cong \text{Alt}(4)$ ; or
    - (3.1.2)  $p$  is odd and  $G = G_0 \ltimes N$  is a Frobenius group with Frobenius kernel  $N$  and Frobenius complement  $G_0$  which is itself an  $\mathfrak{X}$ -group. Furthermore, either
      - (3.1.2.1)  $G_0$  is cyclic of order dividing  $p^2 - 1$  but not dividing  $p - 1$ ;
      - (3.1.2.2)  $G_0$  is quaternion;
      - (3.1.2.3)  $G_0$  is supersoluble as in (2.1.2) with  $|C|$  dividing  $p - \epsilon$  where  $p \equiv \epsilon \pmod{4}$ ;
      - (3.1.2.4)  $G_0$  is supersoluble as in (2.1.3) with  $D$  a 2-group,  $C_D(C)$  a non-trivial maximal subgroup of  $D$  and  $|C|$  odd dividing  $p - 1$  or  $p + 1$ ;
      - (3.1.2.5)  $G_0 \cong \text{SL}_2(3)$ ;
      - (3.1.2.6)  $G_0 \cong \text{SL}_2(3) \cdot 2$  and  $p \equiv \pm 1 \pmod{8}$ ; or
      - (3.1.2.7)  $G_0 \cong \text{SL}_2(5)$  and 60 divides  $p^2 - 1$ .
  - (3.2)  $F^*(G)$  is extraspecial of order  $p^3$  and one of the following holds:
    - (3.2.1)  $G \cong \text{SL}_2(3)$  or  $G \cong \text{SL}_2(3) \cdot 2$  (with quaternion Sylow 2-subgroups of order 16); or
    - (3.2.2)  $G = K \ltimes N$  where  $N$  is extraspecial of order  $p^3$  and exponent  $p$  with  $p$  an odd prime,  $K$  centralizes  $Z(N)$  and is cyclic of odd order dividing  $p + 1$ . Furthermore,  $G/Z(N)$  is a Frobenius group.
- (4) If  $F^*(G)$  is not nilpotent, then either
  - (4.1)  $F^*(G) \cong \text{SL}_2(p)$  where  $p$  is a Fermat prime,  $|G/F^*(G)| \leq 2$  and  $G$  has quaternion Sylow 2-subgroups; or
  - (4.2)  $G \cong \text{PSL}_2(9)$ ,  $\text{Mat}(10)$  or  $\text{PSL}_2(p)$  where  $p$  is a Fermat or Mersenne prime.

Furthermore, all the groups listed above are  $\mathfrak{X}$ -groups.

We make a brief remark about the group  $\text{SL}_2(3) \cdot 2$  and the groups appearing in part (4.1) of Theorem 2.1 in the case  $G > F^*(G)$ . To obtain such groups, take  $F = \text{SL}_2(p^2)$ , then the groups in question are isomorphic to the normalizer in  $F$  of the subgroup isomorphic to  $\text{SL}_2(p)$ . We denote these groups by  $\text{SL}_2(p) \cdot 2$  to indicate that the extension is not split (there are no elements of order 2 in the outer half of the group).

We shall repeatedly use the fact that if  $L$  is a subgroup of an  $\mathfrak{X}$ -group  $X$ , then  $L$  is an  $\mathfrak{X}$ -group. Indeed, if  $H \leq L$  is non-cyclic, then  $C_L(H) \leq C_X(H) \leq H$ .

The following elementary facts will facilitate our proof that the examples listed are indeed  $\mathfrak{X}$ -groups.

**Lemma 2.2.** *The finite group  $X$  is an  $\mathfrak{X}$ -group if and only if  $C_X(x)$  is an  $\mathfrak{X}$ -group for all  $x \in X$  of prime order.*

*Proof.* If  $X$  is an  $\mathfrak{X}$ -group, then, as  $\mathfrak{X}$  is subgroup closed,  $C_X(x)$  is an  $\mathfrak{X}$ -group for all  $x \in X$  of prime order. Conversely, assume that  $C_X(x)$  is an  $\mathfrak{X}$ -group for all  $x \in X$  of prime order (and hence of any order). Let  $H \leq X$  be non-cyclic. We shall show  $C_X(H) \leq H$ . If  $C_X(H) = 1$ , then  $C_X(H) \leq H$  and we are done. So assume  $x \in C_X(H)$  and  $x \neq 1$ . Then  $H \leq C_X(x)$  which is an  $\mathfrak{X}$ -group. Hence  $x \in C_{C_X(x)}(H) \leq H$ . Therefore  $C_X(H) \leq H$ , and  $X$  is an  $\mathfrak{X}$ -group.  $\square$

**Lemma 2.3.** *Suppose that  $X$  is a Frobenius group with kernel  $K$  and complement  $L$ . If  $K$  and  $L$  are  $\mathfrak{X}$ -groups, then  $X$  is an  $\mathfrak{X}$ -group.*

*Proof.* Let  $x \in X$  have prime order. Then, as  $K$  and  $L$  have coprime orders,  $x \in K$  or  $x$  is conjugate to an element of  $L$ . But then, since  $X$  is a Frobenius group, either  $C_X(x) \leq K$  or  $C_X(x)$  is conjugate to a subgroup of  $L$ . Since  $K$  and  $L$  are  $\mathfrak{X}$ -groups,  $C_X(x)$  is an  $\mathfrak{X}$ -group. Hence  $X$  is an  $\mathfrak{X}$ -group by Lemma 2.2.  $\square$

The rest of this section is dedicated to the proof of Theorem 2.1; therefore  $G$  always denotes a finite  $\mathfrak{X}$ -group. Parts (1) and (2) of Theorem 2.1 are already proved in [1, Theorems 2.2, 2.4, 3.2 and 3.4]. However, our statement in (2.1.3) adds further detail which we now explain. So, for a moment, assume that  $G$  is supersoluble,  $q$  is the smallest prime dividing  $|G|$ ,  $D$  is a cyclic  $q$ -group and  $C$  is a cyclic  $q'$ -group. In addition,  $1 \neq Z(G) = C_D(C)$ . Assume that  $d \in D \setminus Z(G)$ . Then, as  $d \notin Z(G)$ ,  $C$  is not centralized by  $d$ . By coprime action,  $C = [C, d] \times C_C(d)$  and so  $Y = [C, d]\langle d \rangle$  is centralized by  $C_C(d)$ . As  $Y$  is non-abelian and  $C_C(d) \cap Y = 1$ , we deduce that  $C_C(d) = 1$ . Hence  $G/Z(G)$  is a Frobenius group. This means that we can assume that (1) and (2) hold and, in particular, we assume that  $G$  is not supersoluble.

The following lemma provides the basic case subdivision of our proof.

**Lemma 2.4.** *One of the following holds:*

- (i)  $F^*(G)$  is elementary abelian of order  $p^2$  for some prime  $p$ .
- (ii)  $F^*(G)$  is extraspecial of order  $p^3$  for some prime  $p$ .
- (iii)  $F^*(G)$  is quasisimple.

*Proof.* Suppose first that  $F^*(G)$  is nilpotent. Then its structure is given in part (1) of Theorem 2.1. Suppose that  $F^*(G)$  is cyclic. Since  $C_G(F^*(G)) = F^*(G)$ , we have  $G/F^*(G)$  is isomorphic to a subgroup of  $\text{Aut}(F^*(G))$ . Because the automorphism group of a cyclic group is abelian, we have that  $G$  is supersoluble. Therefore, by our assumption concerning  $G$ ,  $F^*(G)$  is not cyclic. Hence  $F^*(G)$  is either elementary abelian of order  $p^2$  for some prime  $p$ , is extraspecial of order  $p^3$  for some odd prime  $p$  or  $F^*(G)$  is a dihedral, semidihedral or quaternion 2-group. Since the automorphism groups of dihedral, semidihedral and quaternion groups of order at least 16 are 2-groups, we deduce that when  $p = 2$  and  $F^*(G)$  is non-abelian,  $F^*(G)$  is extraspecial. This proves the lemma when  $F^*(G)$  is nilpotent.

If  $F^*(G)$  is not nilpotent, then there exists a component  $K \leq F^*(G)$ . As  $F^*(G) = C_{F^*(G)}(K)K$  and  $K$  is non-abelian, we have  $F^*(G) = K$  and this is case (iii).  $\square$

**Lemma 2.5.** *Suppose that  $p$  is a prime and  $F^*(G)$  is extraspecial of order  $p^3$ . Then one of the following holds:*

- (i)  $G \cong \text{SL}_2(3)$ ,  $G \cong \text{SL}_2(3) \cdot 2$  (with quaternion Sylow 2-subgroups of order 16); or
- (ii)  $G = NK$  where  $N$  is extraspecial of order  $p^3$  of exponent  $p$  with  $p$  an odd prime,  $K$  centralizes  $Z(N)$  and is cyclic of odd order dividing  $p+1$ . Furthermore,  $G/Z(N)$  is a Frobenius group.

*Proof.* Let  $N = F^*(G)$ . We have that  $N$  is extraspecial of order  $p^3$  by assumption. Suppose first that  $p = 2$ , then we have  $N \cong Q_8$  as the dihedral group of order 8 has no odd order automorphisms and  $G$  is not a 2-group. Since  $\text{Aut}(Q_8) \cong \text{Sym}(4)$ ,  $G/Z(N)$  is isomorphic to a subgroup of  $\text{Sym}(4)$  containing  $\text{Alt}(4)$ . If  $G/Z(N) \cong \text{Alt}(4)$ , then  $G = NT \cong \text{SL}_2(3)$  where  $T$  is a cyclic subgroup of order 3. When  $G/Z(N) \cong \text{Sym}(4)$ , taking  $T \in \text{Syl}_3(G)$ , we have  $NT \cong \text{SL}_2(3)$ ,  $N_G(T)$  has order 12 and  $N_G(T)/Z(N) \cong \text{Sym}(3)$ . Since  $N_G(T)$  is an  $\mathfrak{X}$ -group and  $N_G(T)$  is supersoluble, we see that  $N_G(T)$  is a product  $DT$  where  $D$  is cyclic of order 4 by (2.1.3). Because the Sylow 2-subgroups of  $G$  are either dihedral, semidihedral or quaternion and  $D \not\leq N$ , we see that  $ND$  is quaternion. Thus  $G \cong \text{SL}_2(3) \cdot 2$  as claimed in (i).

Assume that  $p$  is odd. We know that the outer automorphism group of  $N$  is isomorphic to a subgroup of  $\text{GL}_2(p)$  and  $C_{\text{Aut}(N)}(Z(N))/\text{Inn}(N)$  is isomorphic to a subgroup of  $\text{SL}_2(p)$ . Since  $p$  is odd and the Sylow  $p$ -subgroups of  $G$  are  $\mathfrak{X}$ -groups, we have  $N \in \text{Syl}_p(G)$  and  $G/N$  is a  $p'$ -group by part (1) of Theorem 2.1. Set  $Z = Z(N)$ . Since  $G/N$  and  $N$  have coprime orders, the Schur Zassenhaus Theorem says that  $G$  contains a complement  $K$  to  $N$ . Set  $K_1 = C_K(Z)$ . Then  $K_1$  commutes with  $Z$  and so  $K_1$  is cyclic. If  $K_1 = 1$ , then  $|K|$  divides  $p-1$  and we find that  $G$  is supersoluble, which is a contradiction. Hence  $K_1 \neq 1$ . Let  $x \in K_1$ . Then  $[N, x]$  and  $C_N(x)$  commute by the Three Subgroups Lemma. Hence  $C_N(x)$  centralizes  $[N, x]\langle x \rangle$  which is non-abelian. It follows that  $[N, x] = N$  and  $C_N(x) = Z$ . If  $\langle x \rangle$  does not act irreducibly on  $N/Z$ , then there exists  $Z < N_1 < N$  which is  $\langle x \rangle$ -invariant. If  $N_1$  is cyclic, then, as  $\langle x \rangle$  centralizes  $\Omega_1(N_1) = Z$ ,  $\langle x \rangle$  centralizes  $N_1 > Z$ , a contradiction. If  $N_1$  is elementary abelian, then, as  $\langle x \rangle$  centralizes  $Z$ ,  $[N_1, \langle x \rangle]$  has order at most  $p$  by Maschke's Theorem. If  $[N_1, \langle x \rangle] \neq 1$ , then  $[N_1, \langle x \rangle]\langle x \rangle$  is non-abelian and  $Z$  centralizes  $[N_1, \langle x \rangle]\langle x \rangle$ , a contradiction. Hence  $\langle x \rangle$  centralizes  $N_1$  contrary to  $C_N(\langle x \rangle) = Z$ . We conclude that every element of  $K_1$  acts irreducibly on  $N/Z(N)$ . In particular, since  $K_1$  is isomorphic to a subgroup of  $\text{SL}_2(p)$ , we have that  $K_1$  is cyclic of odd order dividing  $p+1$ . Furthermore, as  $K_1$  acts irreducibly on  $N/Z(N)$ ,  $N$  has exponent  $p$ .

By the definition of  $K_1$ ,  $|K/K_1|$  divides  $|\text{Aut}(Z)| = p-1$ . Assume that  $K \neq K_1$  and let  $y \in K \setminus K_1$  have prime order  $r$ . Then  $r$  does not divide  $|K_1|$  and  $Z\langle y \rangle$  is non-abelian. Since  $K_1$  centralizes  $Z$ , we have  $C_{K_1}(y) = 1$ . Let

$w \in K_1$  have prime order  $q$ . Then  $\langle y \rangle \langle w \rangle$  is non-abelian and acts faithfully on  $V = N/Z$ . Therefore [2, 27.18] implies that  $C_N(y) \neq 1$ . As  $C_N(y) \cap Z = 1$  and  $C_N(y)$  centralizes  $Z \langle y \rangle$ , we have a contradiction. Hence  $K = K_1$ . Finally, we note that  $NK/Z(N)$  is a Frobenius group.

It remains to show that the groups listed are  $\mathfrak{X}$ -groups. We consider the groups listed in (ii) and leave the groups in (i) to the reader. Assume that  $H \leq G$  is non-cyclic. We shall show that  $C_G(H) \leq H$ . If  $H \geq N$ , then  $C_G(H) \leq C_G(N) \leq N \leq H$  and we are done. Suppose that  $H < N$ . Then, as  $N$  is extraspecial of exponent  $p$ ,  $H$  is elementary abelian of order  $p^2$  and  $C_N(H) = H$ . Since  $G/N$  is cyclic of odd order dividing  $p+1$ , we see that  $N_G(H) = N$  and so  $C_G(H) = C_N(H) = H$  and we are done in this case. Suppose that  $H \not\leq N$  and  $N \not\leq H$ . Let  $h \in H \setminus N$ . Then, as  $|G/N|$  divides  $p+1$  and is odd, we either have  $H \cap N = N$  or  $H \cap N = Z$ . So we must have  $H \cap N = Z = Z(G)$ . Now  $H/Z \cong G/N$  is cyclic of order dividing  $p+1$  and so we get that  $H$  is cyclic, a contradiction. Thus  $G$  is an  $\mathfrak{X}$ -group.  $\square$

**Lemma 2.6.** *Suppose that  $N = F^*(G)$  is elementary abelian of order  $p^2$ . Then one of the following holds:*

- (i)  $p = 2$ ,  $G \cong \text{Sym}(4)$  or  $\text{Alt}(4)$ ; or
- (ii)  $p$  is odd and  $G = NG_0$  is a Frobenius group with Frobenius kernel  $N$  and Frobenius complement  $G_0$  which is itself an  $\mathfrak{X}$ -group. Furthermore, either
  - (a)  $G_0$  is cyclic of order dividing  $p^2 - 1$  but not dividing  $p - 1$ ;
  - (b)  $G_0$  is quaternion;
  - (c)  $G_0$  is supersoluble as in part (2.1.2) of Theorem 2.1 with  $|C|$  dividing  $p - \epsilon$  where  $p \equiv \epsilon \pmod{4}$ ;
  - (d)  $G_0$  is supersoluble as in part (2.1.3) of Theorem 2.1 with  $D$  a 2-group,  $C_D(C)$  a non-trivial maximal subgroup of  $D$  and  $|C|$  odd dividing  $p-1$  or  $p+1$ ;
  - (e)  $G_0 \cong \text{SL}_2(3)$ ;
  - (f)  $\text{SL}_2(3) \cdot 2$  and  $p \equiv \pm 1 \pmod{8}$ ; or
  - (g)  $G_0 \cong \text{SL}_2(5)$  and 60 divides  $p^2 - 1$ .

Furthermore, all the groups listed are  $\mathfrak{X}$ -groups.

*Proof.* We have  $N$  has order  $p^2$ , is elementary abelian and  $G/N$  is isomorphic to a subgroup of  $\text{GL}_2(p)$ . If  $p = 2$ , then we quickly obtain part (i). So assume that  $p$  is odd.

Suppose that  $p$  divides the order of  $G/N$ . Let  $P \in \text{Syl}_p(G)$ . Then  $P$  is extraspecial of order  $p^3$  and  $P$  is not normal in  $G$ . Hence by [4, Theorem 2.8.4] there exists  $g \in G$  such that  $G \geq K = \langle P, P^g \rangle \cong p^2 : \text{SL}_2(p)$ . Let  $Z = Z(P)$ ,  $t$  be an involution in  $K$ ,  $K_0 = C_K(t)$  and  $P_0 = P \cap K_0$ . Then, as  $t$  inverts  $N$ ,  $K_0 \cong \text{SL}_2(p)$ ,  $P_0$  has order  $p$  and centralizes  $Z \langle t \rangle$ , which is a contradiction as  $Z \langle t \rangle \cong \text{Dih}(2p)$ . Hence  $G/N$  is a  $p'$ -group.

Suppose that  $x \in G \setminus N$ . If  $C_N(x) \neq 1$ , then  $C_N(x)$  centralizes  $[N, x] \langle x \rangle$  which is non-abelian, a contradiction. Thus  $C_N(x) = 1$  for all  $x \in G \setminus N$ . It follows that  $G$  is a Frobenius group with Frobenius kernel  $N$ . Let  $G_0$  be a

Frobenius complement to  $N$ . As  $G_0 \leq G$ ,  $G_0$  is an  $\mathfrak{X}$ -group. Recall that the Sylow 2-subgroups of  $G_0$  are either cyclic or quaternion and that the odd order Sylow subgroups of  $G_0$  are all cyclic [5, V.8.7].

Assume that  $N$  is not a minimal normal subgroup of  $G$ . Then  $G/N$  is conjugate in  $\mathrm{GL}_2(p)$  to a subgroup of the diagonal subgroup. Therefore  $G$  is supersoluble, which is a contradiction. Hence  $N$  is a minimal normal subgroup of  $G$  and  $G_0$  is isomorphic to an irreducible subgroup of  $\mathrm{GL}_2(p)$ . This completes the general description of the structure of  $G$ . It remains to determine the structure of  $G_0$ .

If  $G_0$  is nilpotent, then Theorem 2.1 (1) applies to give  $G_0$  is either quaternion or cyclic. In the latter case, as  $G_0$  acts irreducibly on  $N$  it is isomorphic to a subgroup of the multiplicative group of  $\mathrm{GF}(p^2)$  and is not of order dividing  $p - 1$ . This gives the structures in (ii) (a) and (b).

If  $G_0$  is supersoluble, then the structure of  $G_0$  is described in part (2.1) of Theorem 2.1, as  $\mathrm{GL}_2(p)$  contains no extraspecial subgroups of odd order. We adopt the notation from (2.1). By [5, V.8.18 c)],  $Z(G_0) \neq 1$ . Hence (2.1.1) cannot occur. Case (2.1.2) can occur and, as  $C$  commutes with a non-central cyclic subgroup of order at least 4 and  $G_0$  is isomorphic to a subgroup of  $\mathrm{GL}_2(p)$ ,  $|C|$  divides  $p - 1$  if  $p \equiv 1 \pmod{4}$  and  $|C|$  divides  $p + 1$  if  $p \equiv 3 \pmod{4}$ . In the situation described in part (2.1.3) of Theorem 2.1, the groups have no 2-dimensional faithful representations unless  $q = 2$  and  $C_D(C)$  has index 2. In this case  $|C|$  is an odd divisor of  $p - 1$  or  $p + 1$ .

Suppose that  $G_0$  is not supersoluble. Refereing to Lemma 2.4 and using the fact that the Sylow subgroups of  $G_0$  are either cyclic or quaternion, we have that  $F^*(G_0)$  is either quaternion of order 8 or  $F^*(G_0)$  is quasisimple. In the first case we obtain the structures described in parts (b), (e) and (f) from Lemma 2.5 where for part (f) we note that we require  $\mathrm{SL}_2(p)$  to have order divisible by 16.

If  $F^*(G_0)$  is quasisimple, then Zassenhaus's Theorem [6, Theorem 18.6, p. 204] gives  $G_0 = WM$  where  $W \cong \mathrm{SL}_2(5)$  and  $M$  is metacyclic. Since  $G_0$  is an  $\mathfrak{X}$ -group, this means that  $M \leq W$  and  $G_0 \cong \mathrm{SL}_2(5)$ . Since  $\mathrm{SL}_2(5)$  is isomorphic to a subgroup of  $\mathrm{GL}_2(p)$  only when  $p = 5$  or 60 divides  $p^2 - 1$  and  $p \neq 5$  part (g) holds.

That  $\mathrm{Sym}(4)$  and  $\mathrm{Alt}(4)$  are  $\mathfrak{X}$ -groups is easy to check. The groups listed in (ii) are  $\mathfrak{X}$ -groups by Lemma 2.3.  $\square$

The finite simple  $\mathfrak{X}$ -groups are determined in [1]. We have to extend the arguments to the cases where  $F^*(G)$  is simple or quasisimple. This is relatively elementary.

**Lemma 2.7.** *Suppose that  $F^*(G)$  is simple. Then  $G \cong \mathrm{SL}_2(4)$ ,  $\mathrm{PSL}_2(9)$ ,  $\mathrm{Mat}(10)$  or  $\mathrm{PSL}_2(p)$  where  $p$  is a Fermat or Mersenne prime.*

*Proof.* Set  $H = F^*(G)$ . As  $\mathfrak{X}$  is subgroup closed,  $H$  is an  $\mathfrak{X}$ -group and so  $H$  is one of the groups listed in the statement by Theorem 3.7 of [1]. Hence we obtain  $H \cong \mathrm{SL}_2(4)$ ,  $\mathrm{PSL}_2(9)$  or  $\mathrm{PSL}_2(p)$  for  $p$  a Fermat or Mersenne prime.

Suppose that  $G > H$ . If  $H \cong \mathrm{SL}_2(4)$ , then  $G \cong \mathrm{Sym}(5)$  and the subgroup  $2 \times \mathrm{Sym}(3)$  witnesses the fact that  $\mathrm{Sym}(5)$  is not an  $\mathfrak{X}$ -group. Suppose  $H \cong$



$\text{PSL}_2(9) \cong \text{Alt}(6)$ . If  $G \geq K \cong \text{Sym}(6)$ , then  $G$  contains  $\text{Sym}(5)$  which is impossible. Therefore  $G \cong \text{PGL}_2(9)$  or  $G \cong \text{Mat}(10)$ . In the first case,  $G$  contains a subgroup  $\text{Dih}(20) \cong 2 \times \text{Dih}(10)$  which is impossible. Thus  $G \cong \text{Mat}(10)$  and this group is easily shown to satisfy the hypothesis as all the centralizer of elements of prime order are  $\mathfrak{X}$ -groups.

If  $H \cong \text{PSL}_2(p)$ ,  $p$  a Fermat or Mersenne prime, then  $G \cong \text{PGL}_2(p)$  and contains a dihedral group of order  $2(p+1)$  and one of order  $2(p-1)$ . One of these is not a 2-group and this contradicts  $G$  being an  $\mathfrak{X}$ -group.  $\square$

**Lemma 2.8.** *Suppose that  $F^*(G)$  is quasisimple but not simple. Then  $F^*(G) \cong \text{SL}_2(p)$  where  $p$  is a Fermat prime,  $|G/H| \leq 2$  and  $G$  has quaternion Sylow 2-subgroups.*

*Proof.* Let  $H = F^*(G)$  and  $Z = Z(H)$ . Since  $H$  centralizes  $Z$ , we have  $Z$  is cyclic. Let  $S \in \text{Syl}_2(H)$ . If  $Z \not\leq S$ , then  $S$  must be cyclic. Since groups with a cyclic Sylow 2-subgroup have a normal 2-complement [2, 39.2], this is impossible. Hence  $Z \leq S$ . In particular,  $Z(G) \neq 1$  as the central involution of  $H$  is central in  $G$ . It follows also that all the odd order Sylow subgroups of  $G$  are cyclic. By part (1) of Theorem 2.1,  $S$  is either abelian, dihedral, semidihedral or quaternion. If  $S$  is abelian, then  $S/Z$  is cyclic and again we have a contradiction. So  $S$  is non-abelian. Thus  $S/Z$  is dihedral (including elementary abelian of order 4). Hence  $H/Z \cong \text{Alt}(7)$  or  $\text{PSL}_2(q)$  for some odd prime power  $q$  [4, Theorem 16.3]. Since the odd order Sylow subgroups of  $G$  are cyclic, we deduce that  $H \cong \text{SL}_2(p)$  for some odd prime  $p$ . If  $p-1$  is not a power of 2, then  $H$  has a non-abelian subgroup of order  $pr$  where  $r$  is an odd prime divisor of  $p-1$  which is centralized by  $Z$ . Hence  $p$  is a Fermat prime.

Suppose that  $G > H$  with  $H \cong \text{SL}_2(p)$ ,  $p$  a Fermat prime. Note  $G/H$  has order 2. Let  $S \in \text{Syl}_2(G)$ . Then  $S \cap H$  is a quaternion group. Suppose that  $S$  is not quaternion. Then there is an involution  $t \in S \setminus H$ . By the Baer-Suzuki Theorem, there exists a dihedral group  $D$  of order  $2r$  for some odd prime  $r$  which contains  $t$ . Since  $D$  and  $Z$  commute, this is impossible. Hence  $S$  is quaternion. This gives the structure described in the lemma.

It remains to demonstrate that the groups  $\text{SL}_2(p)$  and  $\text{SL}_2(p) \cdot 2$  with  $p$  a Fermat prime are indeed  $\mathfrak{X}$ -groups. Let  $G$  denote one of these group,  $H = F^*(G) \cong \text{SL}_2(p)$ . Recall from the comments just after the statement of Theorem 2.1 that  $G$  is isomorphic to a subgroup of  $X = \text{SL}_2(p^2)$ . Let  $V$  be the natural  $\text{GF}(p^2)$  representation of  $X$  and thereby a representation of  $G$ . Assume that  $L \leq G$  is non-cyclic. Since  $H$  has no abelian subgroups which are not cyclic,  $L$  is non-abelian and  $L$  acts irreducibly on  $V$ . Schur's Lemma implies that  $C_X(L)$  consists of scalar matrices and so has order at most 2. If  $L$  has even order, then as  $G$  has quaternion Sylow 2-subgroups,  $L \geq C_G(L)$ . So suppose that  $L$  has odd order. Then using Dickson's Theorem [7, 260, page 285], as  $p$  is a Fermat prime, we find that  $L$  is cyclic, a contradiction. Thus  $G$  is an  $\mathfrak{X}$ -group.  $\square$

*Proof of Theorem 2.1.* This follows from the combination of the lemmas in this section.  $\square$



### 3. Infinite locally finite $\mathfrak{X}$ -groups

It has been proved in [1, Theorem 2.2] that an infinite abelian group is in the class  $\mathfrak{X}$  if and only if it is either cyclic or isomorphic to  $\mathbb{Z}_{p^\infty}$  (the Prüfer  $p$ -group) for some prime  $p$ . Moreover, Theorem 2.3 and Theorem 2.5 of [1] imply that every infinite nilpotent  $\mathfrak{X}$ -group is abelian. We start this section by showing that some extensions of infinite abelian  $\mathfrak{X}$ -groups provide further examples of infinite  $\mathfrak{X}$ -groups.

**Lemma 3.1.** *The infinite dihedral group belongs to the class  $\mathfrak{X}$ .*

*Proof.* Write  $G = \langle a, y \mid y^2 = 1, a^y = a^{-1} \rangle$ . Then for every non-cyclic subgroup  $H$  of  $G$  there exist non-zero integers  $n$  and  $m$  such that  $a^n, a^m y \in H$ . It easily follows that  $C_G(H) = 1$ .  $\square$

**Lemma 3.2.** *Let  $G = A\langle y \rangle$  where  $A \cong \mathbb{Z}_{2^\infty}$  and  $\langle y \rangle$  has order 2 or 4, with  $y^2 \in A$  and  $a^y = a^{-1}$ , for all  $a \in A$ . Then  $G$  belongs to the class  $\mathfrak{X}$ .*

*Proof.* It is clear that  $G/A$  has order 2, and  $A$  is the Fitting subgroup of  $G$ . Also  $C_G(A) = A$  and  $Z(G)$  is the subgroup of order 2 of  $A$ . Let  $H$  be a non-cyclic subgroup of  $G$  with  $H \neq A$ . Then  $H \not\leq A$  as every proper subgroup of  $A$  is cyclic. Pick any element  $h \in H \setminus A$ . Then  $G = A\langle h \rangle$  since  $|G : A| = 2$ . Therefore by the Dedekind modular law we get  $H = C\langle h \rangle$ , where  $C = A \cap H > 1$  is finite.

Since  $h = bv$  with  $b \in A$  and  $v \in \langle y \rangle \setminus A$ , we get  $a^h = a^{-1}$  for all  $a \in A$ . In particular,  $C_A(h)$  has order 2 and  $C_G(h)$  has order 4. Since  $C$  has a unique involution and  $h \in C_G(H)$ , we conclude that  $C_G(H) \leq H$  and so  $G$  is an  $\mathfrak{X}$ -group.  $\square$

When  $\langle y \rangle$  has order 2, the group  $G = A \rtimes \langle y \rangle$  of Lemma 3.2 is a generalized dihedral group.

Let  $p$  denote any odd prime. Then, by Hensel's Theorem (see for instance [8, Theorem 127.5]), the group  $\mathbb{Z}_{p^\infty}$  has an automorphism of order  $p-1$ , say  $\phi$ .

**Lemma 3.3.** *The groups  $G = \mathbb{Z}_{p^\infty} \rtimes \langle \phi^j \rangle$  for  $1 \leq j \leq p-1$  are  $\mathfrak{X}$ -groups.*

*Proof.* As  $\mathfrak{X}$  is subgroup closed, it suffices to show that  $G = \mathbb{Z}_{p^\infty} \rtimes \langle \phi \rangle$  is an  $\mathfrak{X}$ -group. Write the elements of  $G$  in the form  $ay$  with  $a \in A \cong \mathbb{Z}_{p^\infty}$  and  $y \in \langle \phi \rangle$ . Suppose there exist non-trivial elements  $a \in A$  and  $y \in \langle \phi \rangle$  such that  $a^y = a$ . For a suitable non-negative integer  $n$ , the element  $a^{p^n}$  has order  $p$  and it is fixed by  $y$ . Then  $y$  centralizes all elements of order  $p$  in  $A$ , and therefore  $y = 1$  by a result due to Baer (see, for instance, [9, Lemma 3.28]). This contradiction shows that  $\langle \phi \rangle$  acts fixed point freely on  $A$ .

Let  $H$  be any non-cyclic subgroup of  $G$ . Then, as  $G/A$  is cyclic,  $A \cap H \neq 1$ . If  $H = A$  then of course  $C_G(H) = H$ . Thus we can assume that there exist non-trivial elements  $a, b \in A$  and  $y \in \langle \phi \rangle$  such that  $a, by \in H$ . Let  $g \in C_G(H)$ . If  $g \in A$  then  $1 = [g, by] = [g, y]$ , so  $g = 1$ . Now let  $g = cz$  with  $c \in A$  and  $1 \neq z \in \langle \phi \rangle$ . Thus  $1 = [cz, a] = [z, a]$ , and  $a = 1$ , a contradiction. Therefore  $C_G(H) \leq H$  for all non-cyclic subgroups  $H$  of  $G$ , so  $G$  is an  $\mathfrak{X}$ -group.  $\square$

**Lemma 3.4.** *An infinite polycyclic group belongs to the class  $\mathfrak{X}$  if and only if it is either cyclic or dihedral.*

*Proof.* Arguing as in the proof of Theorem 3.1 of [1], one can easily prove that every infinite polycyclic  $\mathfrak{X}$ -group is either cyclic or dihedral. On the other hand, the infinite dihedral group belongs to the class  $\mathfrak{X}$  by Lemma 3.1.  $\square$

**Proposition 3.5.** *A torsion-free soluble group belongs to the class  $\mathfrak{X}$  if and only if it is cyclic.*

*Proof.* Let  $G$  be a torsion-free soluble  $\mathfrak{X}$ -group. Then every abelian subgroup of  $G$  is cyclic, so  $G$  satisfies the maximal condition on subgroups by a result due to Mal'cev (see, for instance, [10, 15.2.1]). Thus  $G$  is polycyclic by [10, 5.4.12]. Therefore  $G$  has to be cyclic.  $\square$

In next theorem we determine all infinite soluble  $\mathfrak{X}$ -groups.

**Theorem 3.6.** *Let  $G$  be an infinite soluble group. Then  $G$  is an  $\mathfrak{X}$ -group if and only if one of the following holds:*

- (i)  $G$  is cyclic;
- (ii)  $G \cong \mathbb{Z}_{p^\infty}$  for some prime  $p$ ;
- (iii)  $G$  is dihedral;
- (iv)  $G = A\langle y \rangle$  where  $A \cong \mathbb{Z}_{2^\infty}$  and  $\langle y \rangle$  has order 2 or 4, with  $y^2 \in A$  and  $a^y = a^{-1}$ , for all  $a \in A$ ;
- (v)  $G \cong A \rtimes D$ , where  $A \cong \mathbb{Z}_{p^\infty}$  and  $1 \neq D \leq C_{p-1}$  for some odd prime  $p$ .

*Proof.* First let  $G$  be an  $\mathfrak{X}$ -group. If  $G$  is abelian then (i) or (ii) holds by [1, Theorem 2.2]. Assume  $G$  is non-abelian, and let  $A$  be the Fitting subgroup of  $G$ . Then  $A \neq 1$  and  $C_G(A) \leq A$  as  $G$  is soluble. Let  $N$  be a nilpotent normal subgroup of  $G$ . Then  $N$  is finite, as, otherwise, using  $N$  is self-centralizing and  $G/Z(N)$  is a subgroup of  $\text{Aut}(N)$ , we obtain  $G$  is finite, which is a contradiction. Thus [1, Theorems 2.3 and 2.5] imply that  $N$  is abelian. In particular, as the product of any two normal nilpotent subgroups of  $G$  is again a normal nilpotent subgroup by Fitting's Theorem, we see that the generators of  $A$  commute. Hence  $A$  is abelian. As  $A$  is infinite and abelian,  $A = C_G(A)$  is either infinite cyclic or isomorphic to  $\mathbb{Z}_{p^\infty}$  for some prime  $p$ . In the former case clearly  $G' \leq A$ . In the latter case, let  $C$  be any proper subgroup of  $A$ . Thus  $C$  is finite cyclic. Moreover  $C$  is characteristic in  $A$ , so it is normal in  $G$ , and  $G/C_G(C)$  is abelian since it is isomorphic to a subgroup of  $\text{Aut}(C)$ . It follows that  $G' \leq C_G(C)$ , and again  $G' \leq C_G(A) = A$ . Therefore  $G/A$  is abelian.

If  $A$  is infinite cyclic, then the argument in the proof of Theorem 3.1 of [1] shows that  $G$  is dihedral. Thus (iii) holds.

Let  $A \cong \mathbb{Z}_{p^\infty}$  for some prime  $p$ , and suppose there exists an element  $x \in G$  of infinite order. Then  $x \in G \setminus A$ , and so there exists an element  $y \in A$  such that  $[x, y] \neq 1$ . Then  $\langle y \rangle$  is a finite normal subgroup of  $G$ , so conjugation by  $x$  induces a non-trivial automorphism of  $\langle y \rangle$ . Since  $\text{Aut}(\langle y \rangle)$  is finite, it follows that there is an integer  $n$  such that  $[x^n, y] = 1$ . Now  $y$  is a torsion element and

$x^n$  has infinite order and so  $\langle x^n, y \rangle$  is neither periodic nor torsion-free and this contradicts [1, Theorems 2.2]. Therefore  $G$  is periodic, and  $G/A$  is isomorphic to a periodic subgroup of automorphisms of  $\mathbb{Z}_{p^\infty}$ .

It is well-known that  $\text{Aut}(\mathbb{Z}_{p^\infty})$  is isomorphic to the multiplicative group of all  $p$ -adic units. It follows that periodic automorphisms of  $\mathbb{Z}_{p^\infty}$  form a cyclic group having order 2 if  $p = 2$ , and order  $p - 1$  if  $p$  is odd (see, for instance, [11] for details). In the latter case (v) holds. Finally, let  $p = 2$ . Then  $G/A = \langle yA \rangle$  has order 2, and  $G = A\langle y \rangle$  with  $y \notin A$  and  $y^2 \in A$ . Moreover  $a^y = a^{-1}$ , for all  $a \in A$ . If  $y$  has order 2 then  $G = A \rtimes \langle y \rangle$ . Otherwise from  $y^2 \in A$  it follows  $y^2 = (y^2)^y = y^{-2}$ , so  $y$  has order 4. Therefore  $G$  has the structure described in (iv).

On the other hand, Lemmas 3.1 – 3.3 show that the groups listed in (i) – (v) are  $\mathfrak{X}$ -groups.  $\square$

Finally, we determine all infinite locally finite  $\mathfrak{X}$ -groups.

**Theorem 3.7.** *Let  $G$  be an infinite locally finite group. Then  $G$  is an  $\mathfrak{X}$ -group if and only if one of the following holds:*

- (i)  $G \cong \mathbb{Z}_{p^\infty}$  for some prime  $p$ ;
- (ii)  $G = A\langle y \rangle$  where  $A \cong \mathbb{Z}_{2^\infty}$  and  $\langle y \rangle$  has order 2 or 4, with  $y^2 \in A$  and  $a^y = a^{-1}$ , for all  $a \in A$ ;
- (iii)  $G \cong A \rtimes D$ , where  $A \cong \mathbb{Z}_{p^\infty}$  and  $1 \neq D \leq C_{p-1}$  for some odd prime  $p$ .

*Proof.* Any abelian subgroup of  $G$  is either finite or isomorphic to  $\mathbb{Z}_{p^\infty}$  for some prime  $p$ , so it satisfies the minimal condition on subgroups. Thus  $G$  is a Černikov group by a result due to Šunkov (see, for instance [10, page 436, I]). Hence there exists an abelian normal subgroup  $A$  of  $G$  such that  $A \cong \mathbb{Z}_{p^\infty}$  for some prime  $p$ , and  $G/A$  is finite. It follows that  $G$  is metabelian, arguing as in the proof of Theorem 3.6. Therefore the result follows from Theorem 3.6.  $\square$

Clearly Theorem 2.1 and Theorem 3.7 give a complete classification of locally finite  $\mathfrak{X}$ -groups.

**Corollary 3.8.** *Let  $G$  be an infinite locally nilpotent group. Then  $G$  is an  $\mathfrak{X}$ -group if and only if one of the following holds:*

- (i)  $G$  is cyclic;
- (ii)  $G \cong \mathbb{Z}_{p^\infty}$  for some prime  $p$ ;
- (iii)  $G = A\langle y \rangle$  where  $A \cong \mathbb{Z}_{2^\infty}$  and  $\langle y \rangle$  has order 2 or 4, with  $y^2 \in A$  and  $a^y = a^{-1}$ , for all  $a \in A$ .

*Proof.* Suppose  $G$  is not abelian. Every finitely generated subgroup of  $G$  is nilpotent, so it is either abelian or finite. It easily follows that all torsion-free elements of  $G$  are central. Thus  $G$  is periodic (see [12, Proposition 1]). Therefore  $G$  is direct product of  $p$ -groups (see, for instance, [10, Proposition 12.1.1]). Actually only one prime can occur since  $G$  is an  $\mathfrak{X}$ -group, so  $G$  is a locally finite  $p$ -group. Thus the result follows by Theorem 3.7.  $\square$

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